New series expansions of the ${}_3F_2$ function

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Abstract

We can use the power series definition of ${}_3F_2(a_1,a_2,a_3;b_1,b_2;z)$ to compute this function for z in the unit disk only. In this paper we obtain new expansions of this function that are convergent in larger domains. Some of these expansions involve the polynomial ${}_3F_2(a_1,-n,a_3;b_1,b_2;z)$ evaluated at certain points z. Other expansions involve the Gauss hypergeometric function ${}_2F_1$. The domain of convergence is sometimes a disk, other times a half-plane, other times the region $|z|^2<4|1-z|$. The accuracy of the approximation given by these expansions is illustrated with numerical experiments.

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1 Introduction

The power series definition of the generalized hypergeometric function ${}_3F_2$,

$$
{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\middle|\, z\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n n!}z^n, \tag{1}
$$

with $b_1,b_2\neq -1,-2,\ldots$, converges inside the unit disk. For numerical computations we can use the right hand side of (1) to compute ${}_3F_2$ only in the disk $|z|\leq\rho<1$, with ρ depending on numerical requirements, such as precision and efficiency. To compute the generalized hypergeometric function ${}_3F_2$ outside of that disk we can use Nørlund's formula [7, Eq.(1.21)]:

$$
{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\middle|\, wz\right)=(1-z)^{-a_1}\sum_{k=0}^{\infty}\frac{(a_1)_k}{k!}\,{}_3F_2\left(\begin{matrix}-k,a_2,a_3\\ b_1,b_2\end{matrix}\middle|\, w\right)\left(\frac{z}{z-1}\right)^k, \qquad w\in\mathbb{C}, \tag{2}
$$

valid only for $\Re(z)<1/2$ if $|w-1|<1$. Outside of this region, we can use Bühring's analytic continuation formula [1]: if no two of the numerator parameters a_j in the definition of ${}_3F_2$ differ by an integer, we have that, for $|\mathrm{ph}(w-z)|<\pi$:

$$
{}_3F_2\left(\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\middle|\, z\right)=\prod_{j=1}^{2}\Gamma(b_j)\sum_{k=1}^{3}C_k\,(w-z)^{-a_k}\sum_{n=0}^{\infty}D_n(a_k,w)\,(z-w)^{-n}, \qquad w\in\mathbb{C}, \tag{3}
$$

with

$$C_k := \frac{\prod_{j=1, j\neq k}^{3} \Gamma(a_j - a_k)}{\left(\prod_{j=1, j\neq k}^{3} \Gamma(a_j)\right)\left(\prod_{j=1}^{2} \Gamma(b_j - a_k)\right)}$$

and

$$D_n(a_k, w) := \frac{(a_k)_n \prod_{j=1}^{2}(1+a_k-b_j)_n}{\prod_{j=1}^{3}(1+a_k-a_j)} {}_3F_2\left(\left.\begin{matrix} a_1-a_k-n, a_2-a_k-n, a_3-a_k-n \\ b_1-a_k-n, b_2-a_k-n \end{matrix}\right| w\right).$$

Expansion (3) converges outside the circle $|z-w| = \max\{|w|, |w-1|\}$.

In this paper we investigate new convergent expansions of the generalized hypergeometric function ${}_3F_2$ in larger domains. The starting point is the simple integral representation [6, Eq.(16.5.2)]:

$${}_3F_2\left(\left.\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix}\right| z\right) = \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} {}_2F_1(a_2, a_3, b_2; zt)dt, \tag{4}$$

valid for $\Re(b_1) > \Re(a_1) > 0$, $b_2 \neq 0, -1, -2, ...$, and $|\text{ph}(1-z)| < \pi$. Using the standard integral representation [6, Eq.(15.6.1)] of the Gauss hypergeometric function in (4) we obtain the double integral representation of the ${}_3F_2$ function

$$\begin{aligned} {}_3F_2(a_1, a_2, a_3; b_1, b_2; z) =& \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(a_2)\Gamma(b_2-a_2)} \\ &\times \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}dt \int_0^1 s^{a_2-1}(1-s)^{b_2-a_2-1}(1-zst)^{-a_3}ds, \end{aligned} \tag{5}$$

valid for $\Re(b_1) > \Re(a_1) > 0$, $\Re(b_2) > \Re(a_2) > 0$ and $z \in \mathbb{C}$. If $\Re(a_3) \geq 1$, then $z \notin [1, +\infty)$. When $|z| < 1$, we can expand the integrand in (5) in powers of z and interchange sum and integral: we obtain the series representation (1) of ${}_3F_2\left(\left.\begin{smallmatrix} a_1, a_2, a_3 \\ b_1, b_2 \end{smallmatrix}\right| z\right)$ valid for $|z| < 1$. Therefore, the double integral representation (5) is the analytic continuation in the complex variable z of the ${}_3F_2$ function defined in (1) from the region $|z| < 1$ to the region $z \in \mathbb{C} \setminus [1, +\infty)$. This representation is valid when $|\text{ph}(1-z)| < \pi$; the branch cut chosen for $(1-z)^{-a_3}$ is the negative real axis.

In [5] we have obtained new series expansions for the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ in terms of elementary functions of a, b, c, z; expansions that converge in different domains in the variable z. The technique used in [5] is based on an appropriate expansions of the term $(1-zt)^{-a}$ in the standard integral representation of the ${}_2F_1$ function [6, Eq.(15.6.1)]. In this paper we chase the same objective for the ${}_3F_2$ function applying similar techniques to the integral representations (4) and (5).

When we replace $f(t) = {}_2F_1(a_2, a_3, b_2; zt)$ in the integrand of (4) by the standard Taylor series expansion of $f(t)$ at $t = 0$ and interchange summation and integration, we obtain the power series expansion (1). The Taylor series expansion of $f(t)$ at $t = 0$ is uniformly convergent for any $t \in (0, 1)$ (for any t in the integration domain of (4) or (5)) if $|z| < 1$. Then, the expansion (1) is convergent in the disk $|z| < 1$.

For purposes that will become clearer later, it is more convenient to see the above argument about the region of convergence of the right hand side of (1) from a different point of view, which is the following. The domain of convergence of (1) (the disk $|z| = 1$) is determined by the two following requirements: (i) The interval of integration $(0, 1)$ in (4) must be completely contained in the domain of convergence of the series expansion of $f(t)$, a disk D of center $t = 0$ and radius $r \geq 1$, $D = \{t \in \mathbb{C}, |t| < r\}$. (ii) The branch point $t = 1/z$ of $f(t)$ must be located outside that domain D, which means that z must be located in a region $S =$ *the inverse to the exterior of* D: $S = (D^{\text{EXT}})^{-1} = \{z \in \mathbb{C}, |z| < r^{-1}\}$. Therefore, the smaller D is (the smaller r), the bigger the domain S of validity of (1) is. But D must satisfy (i) and then the smallest possible r is $r = 1$ and $S = \{z \in \mathbb{C}, |z| < 1\}$ (see Figure 1).

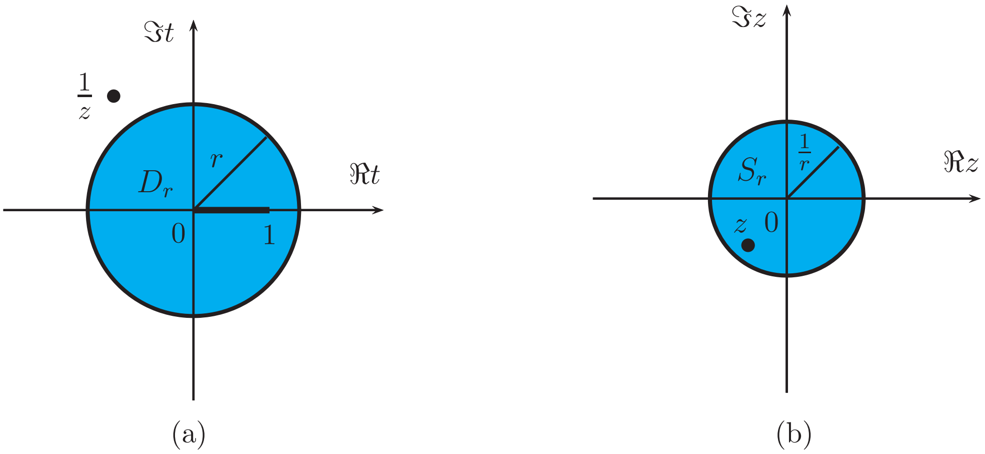


Figure 1: The disk of convergence D of the Taylor expansion of $f(t)$ at $t=0$ is shown in Figure (a) for a certain r (>1), and the region S, inverse of the exterior of D is shown in Figure (b). The smaller D is, the larger S is. The smallest possible value of r for which the integration interval $(0,1)\subset D$ is $r=1$.

In this paper we explore the following idea, used also in [5]. Instead of the Taylor series expansion of $f(t)$ at $t=0$, consider Taylor series expansion of $f(t)$ at different points satisfying the two following requirements:

(i) $(0,1)\subset D$ (The interval of integration $(0,1)$ must be completely contained in D);

(ii) $z\in S:=(D^{\mathrm{EXT}})^{-1}$ (z must be located in a region $S=$ *the inverse to the exterior of* D).

Then, replacing $f(t)={}_2F_1(a_2,a_3,b_2;zt)$ in (4) by this new expansion and interchanging summation and integration, we will obtain an expansion of ${}_3F_2$ convergent for $z\in S$. The larger S is, the better, and one expects that, the smaller D is (containing the interval $(0,1)$ in its interior), the bigger S will be. A similar idea applies when replacing $f(t)=(1-zst)^{-a_3}$ in (5). The first possibility that we explore in Section 2 is an expansion of $f(t)$ $(={}_2F_1(a_2,a_3,b_2;zt)$ in (4) or $=(1-zst)^{-a_3}$ in (5)) at a generic point $t=w$. In Section 3 we consider a two-point Taylor expansion of $f(t)$ at $t=0$ and $t=1$. Some numerical experiments are presented in Section 4. Final remarks and comments are given in Section 5. In the remaining of the paper we assume, without mention, that $b_1,b_2\neq 0,-1,-2,-3,\ldots$.

2 Expansions for $|1-wz|>|z|\max\{|w|,|1-w|\}$ with arbitrary $w\in\mathbb{C}$

2.1 Using the simple integral representation (4)

Consider the Taylor expansion of the function $f(t)={}_2F_1\left(a_2,a_3,b_2;zt\right)$, not at $t=0$, but at a generic point $w=u+vi\neq 0\in\mathbb{C}$, $u,v\in\mathbb{R}$:

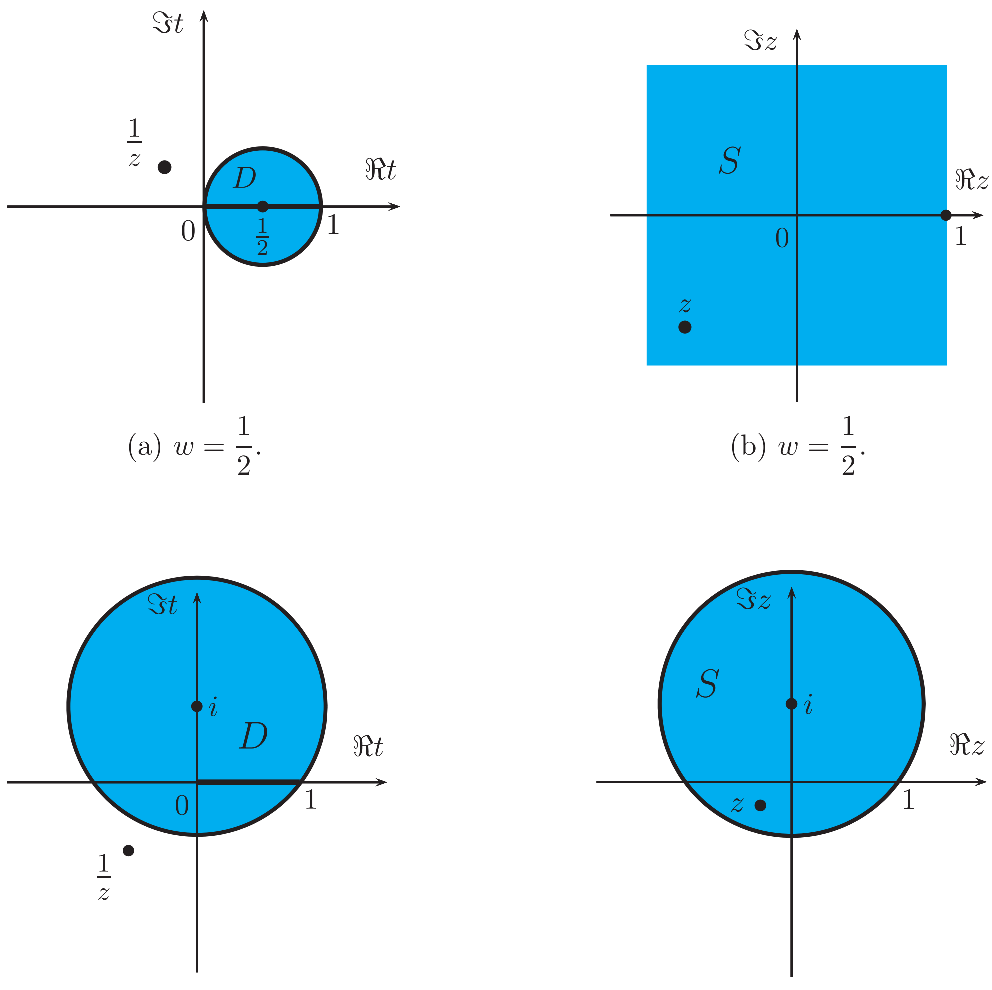
$$f(t)=\sum_{n=0}^{\infty}\frac{(a_2)_n(a_3)_n}{n!(b_2)_n}z^n{}_2F_1\left(\left.\begin{matrix}a_2+n,a_3+n\\ b_2+n\end{matrix}\right|wz\right)(t-w)^n. \tag{6}$$

We exclude $w=0$ because this case corresponds to the series definition (1) of the ${}_3F_2$ function.

Expansion (6) satisfies condition (i) for $D=\{t\in\mathbb{C},\ |t-w|<\max\{|w|,|1-w|\}\}$. It also satisfies condition (ii) (that is, $1/z\notin D$) for $S=\{z\in\mathbb{C},\ |1-wz|>|z|\max\{|w|,|1-w|\}\}$. For $u=\Re w\geq 1/2$ the domain S is the half-plane $S=\{z=x+iy;x,y\in\mathbb{R},\ 2\Re(wz)=2ux-2vy<1\}$. For $u<1/2$ it is the disk $S=\{z\in\mathbb{C},\ |z+(1-2u)^{-1}\overline{w}|<(1-2u)^{-1}|w-1|\}$ (see Figure 2).

Then, for $z\in S$, we can introduce expansion (6) in (4) and interchange summation and integration to obtain

$${}_3F_2\left(\left.\begin{matrix}a_1,a_2,a_3\\ b_1,b_2\end{matrix}\right|z\right)=\sum_{n=0}^{\infty}\frac{(a_2)_n(a_3)_n(-wz)^n}{n!(b_2)_n}\,{}_2F_1\left(\left.\begin{matrix}a_2+n,a_3+n\\ b_2+n\end{matrix}\right|wz\right)\Phi_n(a_1,b_1,w), \tag{7}$$



(a) $w = \frac{1}{2}$. (b) $w = \frac{1}{2}$. (c) $w = i$. (d) $w = i$.

Figure 2: The minimal domain of convergence D of the standard Taylor expansion of $f(t)$ at $t = w$ containing the interval $(0, 1)$ is a disk of center at $t = w$ and radius $\max\{|w|, |1 - w|\}$ (Figures (a) and (c)). The region S, inverse of the exterior of D is: the half-plane $S = \{z = x + iy; x, y \in \mathbb{R},\ 1 - 2\Re(wz) > 0\}$ if $\Re w \geq 1/2$ (Figure (b)) or the disk of center $\overline{w}/(2\Re w - 1)$ and radius $|w - 1|/(1 - 2\Re w)$ if $\Re w < 1/2$ Figure (d)).

with

$$\Phi_n(a_1,b_1,w) := \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)}\int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1}\left(1-\frac{t}{w}\right)^n dt = {}_2F_1\left(\left.\begin{array}{c}-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right).$$

Therefore we get

$$ {}_3F_2\left(\left.\begin{array}{c}a_1,a_2,a_3\\ b_1,b_2\end{array}\right| z\right) = \sum_{n=0}^{\infty}\frac{(a_2)_n(a_3)_n(-wz)^n}{n!(b_2)_n}\,{}_2F_1\left(\left.\begin{array}{c}a_2+n,a_3+n\\ b_2+n\end{array}\right| wz\right){}_2F_1\left(\left.\begin{array}{c}-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right). \tag{8}$$

We have $\Phi_0(a_1,b_1,w)=1$, $\Phi_1(a_1,b_1,w)=1-a_1/(b_1w)$ and, for $n=1,2,3,\ldots$, the remaining $\Phi_n(a_1,b_1,w)$ may be obtained from the three-terms recurrence relation [6, Eq.(15.5.11)]

$$\begin{aligned}(b_1+n)\,{}_2F_1\left(\left.\begin{array}{c}-1-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right) + \left(\frac{a_1+n}{w}-2n-b_1\right){}_2F_1\left(\left.\begin{array}{c}-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right)\\ + n\left(1-\frac{1}{w}\right){}_2F_1\left(\left.\begin{array}{c}1-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right) = 0.\end{aligned}$$

It is straightforward to show that $\Phi_n(a_1,b_1,w)$ also satisfies the contiguous relation

$$\Phi_n(a_1,b_1,w) = \Phi_{n-1}(a_1,b_1,w) - \frac{a_1}{b_1w}\Phi_{n-1}(a_1+1,b_1+1,w).$$

The functions $\Phi_n(a_1,b_1,w)$ are rational functions of a_1 and b_1.

We have derived (8) from (4) by using (6). Therefore, in principle, expansion (8) only converges for $z\in S$ when the integral (4) exists, that is, when $\Re(b_1)>\Re(a_1)>0$. Making a brief analysis, we can extend the domain of validity of that expansion: from [6, Eq.(5.11.7)], [6, Eq.(15.12.3)] and [6, Eq.(15.10.30)], we deduce, respectively,

$$\frac{(a_2)_n(a_3)_n(-wz)^n}{n!(b_2)_n} \sim n^{a_3+a_2-b_2-1}(-wz)^n, \quad n\to\infty,$$

$$ {}_2F_1\left(\left.\begin{array}{c}a_2+n,a_3+n\\ b_2+n\end{array}\right| wz\right) \sim (1-wz)^{-n}, \quad n\to\infty$$

and

$$ {}_2F_1\left(\left.\begin{array}{c}-n,a_1\\ b_1\end{array}\right| \frac{1}{w}\right) \sim A(a_1,b_1,w)n^{-a_1} + B(a_1,b_1,w)\left(\frac{w}{w-1}\right)^{-n} n^{a_1-b_1}, \quad n\to\infty,$$

where A and B are functions of a_1,b_1 and w and are independent of n. Using these equivalences it is straightforward to check that series (8) converges $\forall\, z\in S$ for any value of a_1,a_2,a_3,b_1 and b_2.

2.2 Using the double integral representation (5)

In this section we consider the Taylor expansion of the function $f(st)=(1-zst)^{-a_3}$ in the double integral (5) at a generic point $w=u+vi\neq 0\in\mathbb{C}$, $u,v\in\mathbb{R}$:

$$f(st) = \sum_{n=0}^{\infty}\frac{(a_3)_n z^n}{n!}(1-wz)^{-a_3-n}(st-w)^n. \tag{9}$$

This expansion satisfies condition (i) for $D=\{st\in\mathbb{C},\ |st-w|<\max\{|w|,|1-w|\}\}$. It also satisfies condition (ii) (that is, $1/z\notin D$) for $S=\{z\in\mathbb{C},\ |1-wz|>|z|\max\{|w|,|1-w|\}\}$. For $u=\Re w\geq 1/2$ the domain S is the half-plane $S=\{z=x+iy; x,y\in\mathbb{R},\ 2\Re(wz)=2ux-2vy<1\}$. For $u<1/2$ it is the disk $S=\{z\in\mathbb{C},\ |z+(1-2u)^{-1}\overline{w}|<(1-2u)^{-1}|w-1|\}$ (see Figures 2(b) and 2(d)).

Then, for $z \in S$, we can introduce expansion (9) in (5) and interchange summation and integration to obtain

$$
{}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = (1-wz)^{-a_3} \sum_{n=0}^{\infty} \frac{(a_3)_n}{n!} \left(\frac{wz}{wz-1} \right)^n \Phi_n(a_1, b_1, a_2, b_2, w), \tag{10}
$$

with

$$
\begin{aligned}
\Phi_n(a_1, b_1, a_2, b_2, w) :=& \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(a_2)\Gamma(b_2-a_2)} \\
& \times \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} dt \int_0^1 s^{b-1}(1-s)^{b_2-a_2-1} ds \left(1 - \frac{st}{w}\right)^n dt \\
=& \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1-a_1)} \int_0^1 t^{a_1-1}(1-t)^{b_1-a_1-1} {}_2F_1\left(-n, a_2, b_2; \frac{t}{w}\right) dt \\
=& {}_3F_2\left(\begin{array}{c} a_1, -n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right).
\end{aligned}
$$

Thus, we obtain

$$
{}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = (1-wz)^{-a_3} \sum_{n=0}^{\infty} \frac{(a_3)_n}{n!} \left(\frac{wz}{wz-1} \right)^n {}_3F_2\left(\begin{array}{c} a_1, -n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right). \tag{11}
$$

We have $\Phi_0(a_1, b_1, a_2, b_2, w) = 1$, $\Phi_1(a_1, b_1, a_2, b_2, w) = 1 - a_1 a_2/(b_1 b_2 w)$ and, for $n = 1, 2, 3, \ldots$, the remaining $\Phi_n(a_1, b_1, a_2, b_2, w)$ may be obtained from the four-terms recurret relation [6, Eq.(16.3.6)]

$$
\begin{aligned}
& n(n-1)\left(1 - \frac{1}{w}\right) {}_3F_2\left(\begin{array}{c} a_1, 2-n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right) \\
& - n\left[b_1 + b_2 + 3n - 2 + \frac{1}{w}(1 - 2n - a_1 - a_2)\right] {}_3F_2\left(\begin{array}{c} a_1, 1-n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right) \\
& + \left[(2n+b_1)(2n+b_2) - n - n^2 - \frac{1}{w}(n+a_1)(n+a_2)\right] {}_3F_2\left(\begin{array}{c} a_1, -n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right) \\
& - (n+b_1)(n+b_2)\, {}_3F_2\left(\begin{array}{c} a_1, -1-n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right) = 0.
\end{aligned}
$$

The functions $\Phi_n(a_1, b_1, a_2, b_2)$ are rational functions of a_1, a_2, b_1 and b_2. Observe that formula (11) is Norlund's formula (2).

As in the previous case, in principle, expansion (11) converges for $z \in S$ and values of parameters a_1, a_2, a_3, b_1 and b_2 for which the integral (5) exists, that is, $\Re(b_1) > \Re(a_1) > 0$ and $\Re(b_2) > \Re(a_2) > 0$. But we can extend the domain of convergence of that series: from the integral representation (4) with z replaced by $1/w$ and a_2 replaced by $-n$ and [6, Eq.(5.11.7)], [6, Eq.(15.12.3)], [6, Eq.(15.10.30)] and [6, Eq.(15.10.17)] we find that:

$$
{}_3F_2\left(\begin{array}{c} a_1, -n, a_2 \\ b_1, b_2 \end{array} \middle| \frac{1}{w} \right) \sim \left(A\, n^{-a_2} + B\, n^{-a_1} + C\left(\frac{w-1}{w}\right)^n n^{a_1+a_2-b_1-b_2} \right), \qquad n \to \infty
$$

with A, B and C independent of n. Then, expansion (11) converges $\forall\, z \in S$ for any value of a_1, a_2, a_3, b_1 and b_2.

In any of the above expansions (8) and (11), since we are approximating function $f(t)$ in the interval of integration $(0, 1)$ by means of its Taylor expansion, it should be expected that the optimal choice of w would be $w = 1/2$, halfway point of $(0, 1)$. This value provides a more uniform approximation of the function on the whole interval. Numerical experiments carried out with formulas (8) and (11) suggest the choice of this optimal point.

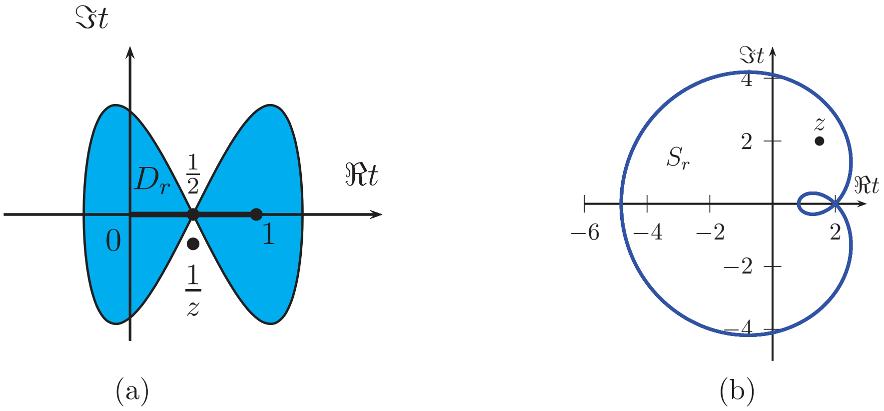


Figure 3: The minimal domain of convergence D of the two-point Taylor expansion of $f(t)$ at $t = 0$ and $t = 1$ containing the interval $(0, 1)$ is a Cassini oval of radius $1/4$ and foci $t = 0$ and $t = 1$ (Figure (a)). The region S, inverse of the exterior of D is the region shown in Figure (b): $S = \{z \in \mathbb{C};\ \ |z|^2 < 4|1 - z|\}$.

3 Expansions for $|z|^2 < 4|1 - z|$

3.1 Using the simple integral representation (4)

As it has been pointed out in [4] (in a different context), the use of a multi-point Taylor expansion [2, 3] with base points in the interval $(0, 1)$ is preferable to using a standard Taylor expansion. With a multi-point Taylor expansion we can avoid the singularity $t = 1/z$ of $f(t)$ in its domain of convergence in a better way and, at the same time, include the whole interval $(0, 1)$ in its interior (see Figure 3(a)).

Therefore, we consider the two-point Taylor expansion of the function $f(t) = {}_2F_1\left(a_2, a_3, b_2; zt\right)$ at $t = 0$ and $t = 1$ [2]:

$$f(t) = \sum_{n=0}^{\infty} [A_n(a_2, a_3, b_2, z) + B_n(a_2, a_3, b_2, z)t]t^n(t - 1)^n. \tag{12}$$

An explicit formula for the coefficients $A_n(a_2, a_3, b_2, z)$ and $B_n(a_2, a_3, b_2, z)$ is given in [2]:

$$A_0(a_2, a_3, b_2, z) = 1, \quad B_0(a_2, a_3, b_2, z) = {}_2F_1\left(a_2, a_3, b_2; z\right) - 1, \tag{13}$$

and for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} A_n(a_2, a_3, b_2, z) &= \sum_{k=0}^{n} \frac{(n+k-1)!(a_2)_{n-k}(a_3)_{n-k}}{n!k!(b_2)_{n-k}(n-k)!} \\ &\quad \times \left[(-1)^n n - (-1)^k k\ {}_2F_1\left(\begin{matrix} a_2+n-k, a_3+n-k \\ b_2+n-k \end{matrix} \middle| z\right)\right] z^{n-k}, \\ B_n(a_2, a_3, b_2, z) &= \sum_{k=0}^{n} \frac{(n+k)!(a_2)_{n-k}(a_3)_{n-k}}{n!k!(b_2)_{n-k}(n-k)!} \left[(-1)^k\, {}_2F_1\left(\begin{matrix} a_2+n-k, a_3+n-k \\ b_2+n-k \end{matrix} \middle| z\right) - (-1)^n\right] z^{n-k}. \end{aligned} \tag{14}$$

To derive these formulas we have used that

$$f^{(n)}(0) = \frac{(a_2)_n(a_3)_n}{(b_2)_n} z^n, \qquad f^{(n)}(1) = \frac{(a_2)_n(a_3)_n}{(b_2)_n} z^n\, {}_2F_1\left(\begin{matrix} a_2+n, a_3+n \\ b_2+n \end{matrix} \middle| z\right).$$

We can also obtain a recurrence relation for these coefficients using the differential equation $t(1 - zt)f'' + [b_2 - (a_2 + a_3 + 1)zt]f' - a_2a_3zf = 0$ satisfied by $f(t) = {}_2F_1(a_2, a_3, b_2; zt)$. Introducing expansion (12) and

$$f'(t) = \sum_{n=0}^{\infty} \{[(2n+1)B_n(a_2,a_3,b_2,z) - (n+1)A_{n+1}(a_2,a_3,b_2,z)] \\ +(n+1)[2A_{n+1}(a_2,a_3,b_2,z) + B_{n+1}(a_2,a_3,b_2,z)]t\}\, t^n(t-1)^n,$$

$$f''(t) = \sum_{n=0}^{\infty} (n+1)\big\{[2(2n+1)A_{n+1}(a_2,a_3,b_2,z) - 2B_{n+1}(a_2,a_3,b_2,z) + (n+2)A_{n+1}(a_2,a_3,b_2,z)] \\ + [2(2n+3)B_{n+1}(a_2,a_3,b_2,z) + (n+2)B_{n+2}(a_2,a_3,b_2,z)]t\big\} t^n(t-1)^n,$$

in the differential equation $t(1-zt)f'' + [b_2 - (a_2+a_3+1)zt]f' - a_2a_3zf = 0$, and equating coefficients of $t^n(t-1)^n$ and $t^{n+1}(t-1)^n$ we obtain the following system of two recursions for the coefficients A_n and B_n:

$$A_n(a_2,a_3,b_2,z) = \frac{1}{n(n+b_2-1)} \\ \times \Bigg\{ \Big[2b_2(1-n) - 6 - 4n^2 + z(1+a_2) + a_3z(1-a_2) + n(10-(a_2+a_3+1)z)\Big] A_{n-1}(a_2,a_3,b_2,z) \\ + \Big[n(2+b_2-3z) + 2(z-1) + n^2z\Big] B_{n-1}(a_2,a_3,b_2,z) \\ + \Big[(2n+a_2-3)z(2n+a_3-3)\Big] B_{n-2}(a_2,a_3,b_2,z)\Bigg\}, \tag{15}$$

$$B_n(a_2,a_3,b_2,z) = \frac{1}{n^2(n-1)(n+b_2-1)(z-1)} \Bigg\{ \Big[n(3-a_2-2n)(a_3+2n-3)z[b_2+(n-1)z]\Big] \\ \times B_{n-2}(a_2,a_3,b_2,z) + (n-n^2)\Big[(2-a_2-2n)(a_3+2n-2)(z-1)z + (6-2b_2+4n(z-1) \\ + (a_2+a_3-5)z)(b_2+(n-1)z)\Big] A_{n-1}(a_2,a_3,b_2,z) + (n-n^2)\Big[b_2(5+6n(z-1) \\ + (a_2+a_3-5)z) - b_2^2 + (n-1)(2+(a_2+a_3-1)z - 2z^2 + n(4z+z^2-4))\Big] \\ \times B_{n-1}(a_2,a_3,b_2,z)\Bigg\}, \tag{16}$$

with

$$\begin{aligned} &A_0(a_2,a_3,b_2,z) = 1, \quad A_1(a_2,a_3,b_2,z) = {}_2F_1(a_2,a_3,b_2;z) - 1 - \frac{a_2a_3z}{b_2}, \\ &B_0(a_2,a_3,b_2,z) = {}_2F_1(a_2,a_3,b_2;z) - 1, \\ &B_1(a_2,a_3,b_2,z) = 2[1 - {}_2F_1(a_2,a_3,b_2;z)] + \frac{a_2a_3z}{b_2}\Big[1 + {}_2F_1(a_2+1,a_3+1,b_2+1;z)\Big]. \end{aligned} \tag{17}$$

Expansion (12) converges inside a Cassini's oval with foci at $t=0$ and $t=1$ and radius $r>0$ of the form $D = \{t \in \mathbb{C},\ |t(t-1)| < r\}$. The interval $(0,1)$ is completely contained in this Cassini oval if its middle point $t_0 = 1/2$ is contained. This happens for $r \geq t_0^2 = 1/4$ and then, expansion (12) satisfies condition (i) for $r \geq 1/4$. On the other hand, it satisfies condition (ii) if $1/z \notin D$ [2], that is, for any

$$r < \left|\frac{1}{z}\left(\frac{1}{z} - 1\right)\right|.$$

The smallest r we can take is $r = 1/4$ and then, the largest S we can choose is (see Figure 3(b))

$$S = \{z \in \mathbb{C};\ |z|^2 < 4|1-z|\} = \{x+iy; x,y \in \mathbb{R},\ y^4 + (2x^2-16)y^2 + [x^4 - 16x^2 + 32x - 16] < 0\}. \tag{18}$$

Then, for $z \in S$, we can introduce expansion (12) in (4) and interchange summation and integration to obtain

$$ {}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z\right) = \sum_{n=0}^{\infty} \left[A_n(a_2, a_3, b_2, z)\Phi_n(a_1, b_1) + B_n(a_2, a_3, b_2, z)\Psi_n(a_1, b_1)\right], \tag{19} $$

with

$$ \Phi_n(a_1, b_1) := (-1)^n \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \int_0^1 t^{a_1+n-1}(1-t)^{n+b_1-a_1-1} dt = (-1)^n \frac{(a_1)_n (b_1 - a_1)_n}{(b_1)_{2n}}, $$

$$ \Psi_n(a_1, b_1) := (-1)^n \frac{\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1 - a_1)} \int_0^1 t^{a_1+n}(1-t)^{n+b_1-a_1-1} dt = (-1)^n \frac{(a_1)_{n+1} (b_1 - a_1)_n}{(b_1)_{2n+1}}. $$

Therefore, we have

$$ {}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(a_1)_n (b_1 - a_1)_n}{(b_1)_{2n+1}} \left[(b_1 + 2n)A_n(a_2, a_3, b_2, z) + (a_1 + n)B_n(a_2, a_3, b_2, z)\right], \tag{20} $$

with $A_n(a_2, a_3, b_2, z)$ and $B_n(a_2, a_3, b_2, z)$ given in (15) and (17).

In principle, expansion (20) converges for $z \in S$ and values of parameters a_1, a_2, a_3, b_1 and b_2 for which the integral (4) exists, that is, $\Re(b_1) > \Re(a_1) > 0$. But once again, we can extend the domain of validity of that series: from the Cauchy's integral representations of A_n and B_n given in [2] and [6, Eq.(5.11.7)], we can see that, when $n \to \infty$, the general term in (20) is behaves as $n^{-1/2} \left|z^2/(4(1-z))\right|^n$. Therefore, formula (20) converges $\forall\, z \in S$ for any value of a_1, a_2, a_3, b_1 and b_2.

3.2 Using the double integral representation (5)

We consider in this section the two-point Taylor expansion of the function $f(st) = (1 - zst)^{-a_3}$ at $st = 0$ and $st = 1$ [2]:

$$ f(st) = \sum_{n=0}^{\infty} [A_n(a_3, z) + B_n(a_3, z)st](st)^n (st - 1)^n. \tag{21} $$

Once again, we can use the explicit formula given in [2] for the coefficients $A_n(a_3, z)$ and $B_n(a_3, z)$:

$$ A_0(a_3, z) = 1, \qquad B_0(a_3, z) = (1 - z)^{-a_3} - 1 $$

and, for $n = 1, 2, 3, \ldots$,

$$ A_n(a_3, z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(n + k - 1)!}{k!(n - k)!} [(-1)^n n - (-1)^k k(1 - z)^{k - a_3 - n}](a_3)_{n-k} z^{n-k}, $$

$$ B_n(a_3, z) = \frac{1}{n!} \sum_{k=0}^{n} \frac{(n + k)!}{k!(n - k)!} [(-1)^k (1 - z)^{k - a_3 - n} + (-1)^{n+1}](a_3)_{n-k} z^{n-k}. $$

Also, a recurrent relation for $A_n(a_3, z)$ and $B_n(a_3, z)$ may be obtained by using the differential equation $(1 - zu)f'(u) = a_3 z f(u)$ satisfied by $f(u)$. Introducing expansion (21) and

$$ f'(u) = \sum_{n=0}^{\infty} \{[(2n+1)B_n(a_3, z) - (n+1)A_{n+1}(a_3, z)] + (n+1)(2A_{n+1}(a_3, z) + B_{n+1}(a_3, z))u\} u^n (u - 1)^n, $$

in the differential equation $(1-zu)f'(u) = a_3 z f(u)$, and equating coefficients of $u^n(u-1)^n$ and $u^{n+1}(u-1)^n$ we obtain:

$$\begin{aligned} A_{n+1}(a_3,z) =& \frac{-z(a_3+2n)A_n(a_3,z)+[1+n(2-z)]B_n(a_3,z)}{n+1}, \\ B_{n+1}(a_3,z) =& \frac{z(2-z)(a_3+2n)A_n(a_3,z)+[z(a_3+2)+n(6z-z^2-4)-2]B_n(a_3,z)}{(n+1)(1-z)}. \end{aligned} \tag{22}$$

Expansion (21) for $f(u)$ converges inside the same Cassini's oval than expansion (12). Therefore, we can introduce expansion (21) in (5) and interchange summation and integration for $z \in S$, where S is the same region (18) considered in Section 3.1. We obtain

$${}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = \sum_{n=0}^{\infty} \left[A_n(a_3,z)\Phi_n(a_1,b_1,a_2,b_2) + B_n(a_3,z)\Psi_n(a_1,b_1,a_2,b_2) \right], \tag{23}$$

with

$$\begin{aligned} \Phi_n(a_1,b_1,a_2,b_2) :=& \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(a_2)\Gamma(b_2-a_2)} \\ & \times \int_0^1 t^{n+a_1-1}(1-t)^{b_1-a_1-1}dt \int_0^1 s^{n+a_2-1}(1-s)^{b_2-a_2-1}(st-1)^n ds \\ =& \frac{(-1)^n(a_1)_n(a_2)_n}{(b_1)_n(b_2)_{n+1}} \\ & \times \left[(b_2-a_2)\, {}_3F_2\left(\begin{array}{c} a_1+n, -n, a_2+n \\ b_1+n, b_2+n+1 \end{array} \middle| 1 \right) + (a_2+n)\, {}_3F_2\left(\begin{array}{c} a_1+n, -n, a_2+n+1 \\ b_1+n, b_2+n+1 \end{array} \middle| z \right) \right], \end{aligned} \tag{24}$$

and

$$\begin{aligned} \Psi_n(a_1,b_1,a_2,b_2) :=& \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(b_1-a_1)\Gamma(a_2)\Gamma(b_2-a_2)} \\ & \times \int_0^1 t^{n+a_1}(1-t)^{b_1-a_1-1}dt \int_0^1 s^{n+a_2}(1-s)^{b_2-a_2-1}(st-1)^n ds \\ =& \frac{a_1 a_2}{b_1 b_2} \frac{\Gamma(b_1+1)\Gamma(b_2+1)}{\Gamma(a_1+1)\Gamma(b_1-a_1)\Gamma(a_2+1)\Gamma(b_2-a_2)} \\ & \times \int_0^1 t^{n+a_1}(1-t)^{b_1-a_1-1}dt \int_0^1 s^{n+a_2}(1-s)^{b_2-a_2-1}(st-1)^n ds \\ =& \frac{a_1 a_2}{b_1 b_2} \Phi_n(a_1+1, b_1+1, a_2+1, b_2+1). \end{aligned}$$

Therefore,

$${}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = \sum_{n=0}^{\infty} \left[A_n(a_3,z)\Phi_n(a_1,b_1,a_2,b_2) + \frac{a_1 a_2}{b_1 b_2} B_n(a_3,z)\Phi_n(a_1+1,b_1+1,a_2+1,b_2+1) \right], \tag{25}$$

with $\Phi_n(a_1,b_1,a_2,b_2)$ given in (24), $A_n(a_3,z)$ and $B_n(a_3,z)$ given by the recursion (22) and $A_0(a_3,z) = 1$, $B_0(a_3,z) = (1-z)^{-a_3} - 1$. This expansion is a series of elementary functions of z: a linear combination of 1 and $(1-z)^{-n-a_3}$ whose coefficients are polynomials in z.

In principle, expansion (25) converges for $z \in S$ and for the values of the parameters a_1, a_2, a_3, b_1 and b_2 for which the integral (5) exists, that is, $\Re(b_1) > \Re(a_1) > 0$ and $\Re(b_2) > \Re(a_2) > 0$. Once again we can extend the domain of validity of that series: using the Cauchy integral representation of A_n and B_n given in [2] and $|\Phi_n(a,b,c,d)| \le 4^{-n}$, we find that the general term in (25) behaves, when $n \to \infty$, as $\left|z^2/(4(1-z))\right|^n$. Therefore, expansion (25) converges $\forall\, z \in S$ for any value of a_1, a_2, a_3, b_1 and b_2.

4 Numerical experiments

The following tables illustrate the approximation supplied by the expansions given in Sections 2 and 3 for ${}_3F_2(a_1, a_2, a_3; b_1, b_2; z)$ for different values of the parameters and the variable. We compare the accuracy, computation time and number of terms required to get a certain precision for the different available approximations: (i) definition (1), (ii) Bühring's expansion (3) and (iii) expansions (8), (11), (20) and (25).

In all of these tables, the first row represents the number of terms n used in each expansion and the following rows represent the relative error obtained with the different expansions considered. The last two columns represent the number of terms and computation time needed to obtain the required precision (relative error smaller than 10^{-16}).

Numerical experiments suggest that approximations derived from a two-point Taylor expansion ((20) and (25)) are more accurate than approximations derived from a standard Taylor expansion ((8) and (11)): two-point Taylor expansions provide more uniform approximations than Taylor series.

With respect to the approximations obtained from the simple integral representation (4) (expansions (8) and (20)) and the ones obtained from the double integral representation (5) (expansions (11) and (25)), the first ones seem to be more efficient than the second ones.

Series	Point w	0	2	4	6	8	R.E.$\leq e{-16}$	Time (sec)
Definition	0	0.137	0.005	0.00024	0.000014	$8.6e{-7}$	26	0.281
Bühring	1/2	0.285	0.196	0.1574	0.0973	0.0545	> 100	+100
Formula (8)	1/2	0.0041	0.000041	$4.8e{-7}$	$6.1e{-9}$	$8.2e{-11}$	16	0.313
Formula (8)	2/3	0.03708	0.00058	0.000011	$2e{-7}$	$4.1e{-9}$	18	0.375
Formula (11)	1/2	0.0441	0.00056	$7.6e{-6}$	$1.1e{-7}$	$1.5e{-9}$	16	0.203
Formula (11)	2/3	0.0956	0.001923	0.000042	$9.6e{-7}$	$2.3e{-8}$	19	0.25
Formula (20)	0, 1	0.0096	$1.4e{-6}$	$2.8e{-10}$	$6.1e{-14}$	$1.23e{-16}$	8	0.515
Formula (25)	0, 1	0.016	$3.9e{-6}$	$9.9e{-10}$	$2.5e{-13}$	$1.6e{-16}$	8	0.266

Table 1: $a_1 = 1$, $a_2 = 1.3$, $a_3 = 1.6$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -\frac{1+i}{5}$

Series	Point w	0	2	4	6	8	R.E.$\leq e{-16}$	Time (sec)
Definition	0	0.0848	0.00194	0.000074	$3.5e{-6}$	$1.8e{-7}$	23	0.297
Bühring	1/2	1.286	1.231	0.7473	0.4085	0.2134	> 100	+100
Formula (8)	1/2	0.0026	0.000016	$1.5e{-7}$	$1.62e{-9}$	$1.89e{-11}$	14	0.328
Formula (8)	2/3	0.0181	0.00015	$1.8e{-6}$	$2.8e{-8}$	$4.8e{-10}$	16	0.39
Formula (11)	1/2	0.0642	0.00073	$9.4e{-6}$	$1.27e{-7}$	$1.8e{-9}$	16	0.188
Formula (11)	2/3	0.1081	0.001991	0.000042	$1.1e{-6}$	$2.2e{-8}$	19	0.25
Formula (20)	0, 1	0.00465	$4.16e{-7}$	$6.08e{-11}$	$1.07e{-14}$	$7.37e{-18}$	7	0.921
Formula (25)	0, 1	0.0111	$2.11e{-6}$	$4.7e{-10}$	$1.12e{-13}$	$1.18e{-16}$	8	0.344

Table 2: $a_1 = 1$, $a_2 = 1.01$, $a_3 = 1.02$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -\frac{1+i}{5}$

Series	Point w	0	2	4	6	8	R.E.$\leq e{-16}$	Time (sec)
Bühring	0	0.2064	0.0215	0.0061	0.0021	0.00078	86	5.18
Bühring	1/2	0.2425	0.0127	0.00086	0.000067	$5.6e{-6}$	28	0.948
Formula (8)	1/2	0.0473	0.00517	0.000686	0.000101	0.000015	38	0.875
Formula (11)	1/2	0.20	0.0316	0.00518	0.000889	0.000157	42	0.344
Formula (20)	0, 1	0.1342	0.003834	0.000138	$5.4e{-6}$	$2.28e{-7}$	22	4.703
Formula (25)	0, 1	0.2101	0.0092	0.000424	0.00002	$9{,}5e{-7}$	24	2.297

Table 3: $a_1 = 1$, $a_2 = 1.3$, $a_3 = 1.6$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -(1+i)$

Series	Point w	0	2	4	6	8	R.E.$\leq e{-16}$	Time (sec)
Bühring	0	0.5257	0.0303	0.00545	0.00139	0.00042	78	6.313
Bühring	1/2	0.2648	0.0477	0.00488	0.00044	0.000038	30	1.5
Formula (8)	1/2	0.01901	0.001529	0.000163	0.0000203	$2.76e{-6}$	36	1.125
Formula (11)	1/2	0.2748	0.03948	0.006297	0.001059	0.0001843	42	0.359
Formula (20)	0, 1	0.05924	0.00922	0.000024	$7.64e{-7}$	$2.69e{-8}$	20	9.188
Formula (25)	0, 1	0.1338	0.00437	0.00176	$7.6e{-6}$	$3.43e{-7}$	23	3.515

Table 4: $a_1 = 1$, $a_2 = 1.01$, $a_3 = 1.02$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -(1+i)$

When we take values of z close to 0, our 4 new approximations are more competitive than Bühring's formula or the definition expansion. Also, when the difference between any couple of parameters a_j is close to an integer (see Tables 2 or 4), our 4 approximations are more competitive than Bühring's approximation. Also, we can observe that formula (11) is the fastest one.

Series	Point w	0	3	6	9	12	R.E.$\leq e{-16}$	Time (sec)
Bühring	0	0.07661	0.000195	$2.6e{-6}$	$4.9e{-8}$	$1.1e{-9}$	7	0.047
Bühring	1/2	0.1323	0.000229	$4.8e{-7}$	$1.1e{-9}$	$2.3e{-12}$	6	0.063
Formula (8)	1/2	0.1121	0.03221	0.00192	0.000773	0.000059	20	0.281
Formula (11)	1/2	0.3471	0.06549	0.01083	0.00235	0.000452	25	0.156
Formula (20)	0, 1	0.3595	0.03738	0.00565	0.000971	0.000177	22	4.485
Formula (25)	0, 1	0.53401	0.09563	0.01916	0.003983	0.000843	26	2.219

Table 5: $a_1 = 1$, $a_2 = 1.3$, $a_3 = 1.6$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -3 + i$

Series	Point w	0	3	6	9	12	R.E.$\leq e{-16}$	Time (sec)
Bühring	0	$--$	$--$	$--$	$--$	$--$	$--$	$--$
Bühring	1/2	$--$	$--$	$--$	$--$	$--$	$--$	$--$
Formula (8)	1/2	0.57473	0.19153	0.0641	0.02189	0.00756	26	0.437
Formula (11)	1/2	0.65835	0.24351	0.08766	0.03154	0.01134	28	0.156
Formula (20)	0, 1	0.30297	0.12404	0.01362	0.00631	0.00068	30	10.719
Formula (25)	0, 1	0.41212	0.14567	0.02429	0.008206	0.001398	32	1.109

Table 6: $a_1 = 1$, $a_2 = 2$, $a_3 = 2$, $b_1 = 1.9$, $b_2 = 2.2$ and $z = -3 + i$

When z is away from 0, Bühring's approximation seems to be more competitive than ours (except when two numerator parameters a_j differ by an integer (see Table 6), since in this case Bühring's formula can't be used). In this case also, formula (11) is faster.

5 Concluding remarks

In Section 2.1 we have used a standard one-point Taylor expansion of $f(t) = {}_2F_1(a_2, a_3, b_2; zt)$ at an arbitrary point $t = w$. Similarly, in Section 2.2 for the function $f(t) = (1 - zt)^{-a_3}$. In Sections 3.1 and 3.2 we have used a two-point Taylor expansion of $f(t) = {}_2F_1(a_2, a_3, b_2; zt)$ and $f(t) = (1 - zt)^{-a_3}$ respectively. One may consider the possibility of expanding $f(t)$ in (4) or in (5) at three or more points . In fact, one gets new approximations valid in larger regions S than the ones shown in Figure 3(b); but those approximations become more and more complicated.

The technique exploited in this paper can be a powerful tool to get new series expansions of other special functions.

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